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Second-Order Global Optimality Conditions for Optimization Problems

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Abstract. Second-order optimality conditions are studied for the constrained optimization problem where the objective function and the constraints are compositions of convex functions and twice strictly differentiable functions. A second-order sufficient condition of a global minimizer is obtained by introducing a generalized representation condition. Second-order minimizer characterizations for a convex program and a linear fractional program are derived using the generalized representation condition.

Key words: Convex program, Global optimality, Linear fractional program, Second-order derivative, Second-order solution characterization

1. Introduction

It is well known that the convex composite model problem includes most of nonlinear optimization problems in the literature, see Burke and Poliqun (1992), Ioffe (1979), Jeyakumar and Yang (1993) and Rockafellar (1988). Second-order sufficient conditions of a strict local minimizer for convex composite problems have been given in Burke and Poliqun (1992) and Rockafellar (1988) by enforcing the inequality in the necessary condition part to be a strict inequality on a larger critical direction set. It has been shown that second-order sufficient conditions of a strict local minimizer for nonlinear programming problems are useful in establishing convergence properties of nonlinear programming algorithms.

The optimality conditions for a global minimizer is important, e.g., in nonconvex (concave) optimization, see Horst and Tuy (1990). It is known that for a convex optimization problem any stationary point is also a global minimizer. Some first-order global optimality conditions for (multi-objective) convex composite problems have been given in Jeyakumar and Yang (1993) by virtue of a (first-order) representation condition. The question is how to characterize the second-order global sufficient condition of a minimizer for a nonconvex optimization problem.

Recently a second-order global sufficient condition has been established for a convex composite optimization problem with an extended real-valued convex function in Yang (1998).

In this paper, second-order optimality conditions are studied for the constrained convex composite optimization problem where the objective function and constraints are compositions of convex functions and twice strictly differentiable functions. A second-order sufficient condition of a global minimizer for a constrained convex composite optimization problem is given by using a generalized representation condition. We show that this generalized representation condition is also useful in characterizing the minimizer sets for a convex program and a linear fractional program. In particular, for a convex quadratic program, we obtain an equivalent condition to the one given in Burke and Ferris (1993).

The outline of the paper is as follows. In Section 2, second-order optimality conditions for a convex composite optimization problem is discussed and a generalized representation condition is also presented. In Section 3, a second-order global sufficient condition for a constrained convex composite optimization problem is given. In Sections 4 and 5, second-order minimizer characterizations for a convex program and a linear fractional program are derived using the generalized representation condition, respectively.

2. Preliminary results

Let $\mathbb{R}^n$ denote an n-dimensional space, and $\langle u, v\rangle$ denote the inner product of vectors $u, v \in \mathbb{R}^n$. Let $g : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous convex function. The convex conjugate of g is defined by

$$g^*(y^*) = \sup\{\langle y^*, y\rangle - g(y) : y \in \mathbb{R}^n\}, \qquad y^* \in \mathbb{R}^n,$$

thus

$$\langle y^*, y\rangle \leqslant g(y) + g^*(y^*), \qquad \forall y \in dom(g), y^* \in \mathbb{R}^n,$$

where $dom(g) = \{y \in \mathbb{R}^m : g(y) < +\infty\}$. The convex subdifferential of g at $y \in dom(g)$ is defined by

$$\partial g(y) = \{y^* \in \mathbb{R}^n : (y^*, -1) \in N(y, g(y)|epi(g))\},$$

where $epi(g)$ is the epi-graph of g, i.e.,

$$epi(g) = \{(y, \alpha) \in \mathbb{R}^n \times \mathbb{R} : g(y) \leqslant \alpha\},$$

and the *normal cone* to convex subset $epi(g)$ of $\mathbb{R}^n$ at $(y, g(y))$ is defined by

$$N((y, g(y))|epi(g)) = \{z^* \in \mathbb{R}^n \times \mathbb{R} : \langle z^*, z - (y, g(y))\rangle \leqslant 0, \forall z \in epi(g)\}.$$

If g is finite at y and $y^* \in \partial g(y)$, then

$$g(z) \geqslant g(y) + \langle y^*, z - y \rangle, \qquad \forall z \in dom(g).$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz function and let $x, u, v \in \mathbb{R}^n$. The *(Clarke) generalized second-order directional derivative* of f at x in the directions (u, v) and the *generalized Hessian* of f at x with respect to u is defined in Cominetti and Correa (1990), respectively, by

$$f^{\circ\circ}(x; u, v) = \limsup_{y \to x, s, t \downarrow 0} \frac{f(y + su + tv) - f(y + su) - f(y + tv) + f(y)}{st},$$
$$\partial^2 f(x)(u) = \{x^* \in \mathbb{R}^n : f^{\circ\circ}(x; u, v) \geqslant \langle x^*, v \rangle, \forall v \in \mathbb{R}^n\}.$$

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be $C^{1,1}$ if it is continuously differentiable with a locally Lipschitz gradient. See Hiriart-Urruty et al (1984) and Yang and Jeyakumar (1992).

The function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *twice strictly differentiable* at $x \in \mathbb{R}^n$ if there exists a linear operator $D^2 f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\lim_{y \to x, s, t \downarrow 0} \frac{f(y + su + tv) - f(y + su) - f(y + tv) + f(y)}{st} = \langle D^2 f(x) u, v \rangle,$$

for all $u, v \in \mathbb{R}^n$. It is clear that the correspondence between linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $n \times n$ symmetric matrices is one-to-one, see Hiriart-Urruty et al (1984). Thus second-order strict derivative $D^2 f(x)$ of f at x is an $n \times n$ symmetric matrix. $F = (F_1, \cdots, F_m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be twice strictly differentiable at x if each component F_i is twice strictly differentiable at x. All linear and quadratic functions are twice strictly differentiable. A twice strictly differentiable function is $C^{1,1}$. It is clear that the generalized Hessian $\partial^2 f(x)(u)$ is singleton for each $u \in \mathbb{R}^n$ if and only if f is twice strictly differentiable at x.

Consider the convex composite optimization problem

$$\text{(CP)} \qquad \text{Minimize } f(x) \quad \text{subjectto } x \in \mathbb{R}^n,$$

where $f(x) = g(F(x))$, $g : \mathbb{R}^m \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous convex function, and $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuously differentiable. If F is twice strictly differentiable near a given point, the Jacobian of F at x, $\nabla F(x)$ is an $m \times n$ matrix, second strict derivative $D^2 F_i(x)$ is an $n \times n$ matrix for each $i = 1, \cdots, m$ and $D^2 F(x) = (D^2 F_1(x)^T, \cdots, D^2 F_m(x)^T)^T$.

As in Burke and Poliquin (1992), let

$$L(x, y^*) = \langle y^*, F(x)\rangle - g^*(y^*), \qquad y^* \in dom(g^*),$$
$$K(x) = \{u \in \mathbb{R}^n : g(F(x) + t\nabla F(x)u) \leqslant g(F(x)), \text{forsome } t > 0\},$$
$$L_0(x) = \{y^* \in \mathbb{R}^m : y^* \in \partial g(F(x)), \nabla F(x)^T y^* = 0\}.$$

$L_0(x)$ is known as the set of optimal multipliers and $K(x)$ is the critical cone. Note that $L(x, y^*)$ is continuously differentiable as a function of x.

The function $f(x) = g(F(x))$ is said to satisfy the **basic constraint qualification** at a point x satisfying $F(x) \in dom(g)$ (Rockafellar (1988)) if the only point $w \in N(F(x)|dom(g))$ for which $0 \in w^T\partial F(x)$ is $w = 0$, and $\partial F(x)$ is the generalized Jacobian of F at x, see Clarke (1983). We see that $L_0(\overline{x})$ is compact and that if $\overline{x}$ satisfying $F(\overline{x}) \in dom(g)$ is a local minimizer of (CP) at which the basic constraint qualification holds, then $L_0(\overline{x}) \neq \emptyset$ (see Burke and Poliquin (1992)).

The following are second-order necessary conditions for (CP). See Yang (1998).

THEOREM 2.1. *Consider the problem* (*CP*). *Let* $F(\overline{x}) \in dom(g)$. *If* F *is continuously differentiable and* $\overline{x}$ *is a local minimizer of* (*CP*) *at which the basic constraint qualification holds, then* $L_0(\overline{x}) \neq \emptyset$ *and*

$$\max\{L^{\circ\circ}(\overline{x}, y^*; u, u) : y^* \in L_0(\overline{x})\} \geqslant 0, \qquad \forall u \in \overline{K(\overline{x})}. \tag{1}$$

Furthermore if F *is twice strictly differentiable at* $\overline{x}$, *then*

$$\max\{\sum_{i=1}^{n} y_i^* u^T D^2 F_i(\overline{x})u : y^* \in L_0(\overline{x})\} \geqslant 0, \qquad \forall u \in \overline{K(\overline{x})}. \tag{2}$$

The following generalized representation condition can be used to establish a second-order sufficient condition of a global minimizer.

DEFINITION 2.1. *Let* $M \subset \mathbb{R}^n$ *be a set and* $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ *be twice strictly differentiable at* $\overline{x}$. *We say that a* **generalized representation condition** *holds for* F *at* $\overline{x}$ *with respect to* M *if for every* $x \in R^n$, *there exists* $\eta(x, \overline{x}) \in M$ *such that*

$$F(x) = F(\overline{x}) + \nabla F(\overline{x})(x - \overline{x}) + \frac{1}{2}\eta(x, \overline{x})^T D^2 F(\overline{x})\eta(x, \overline{x}) \tag{3}$$

where

$$\eta(x, \overline{x})^T D^2 F(\overline{x})\eta(x, \overline{x}) = \begin{pmatrix} \eta(x, \overline{x})^T D^2 F_1(\overline{x})\eta(x, \overline{x}) \\ \vdots \\ \eta(x, \overline{x})^T D^2 F_k(\overline{x})\eta(x, \overline{x}) \end{pmatrix}.$$

PROPOSITION 2.1. *Let $M_1 \subset M_2$. If (3) holds for F at $\overline{x}$ with respect to M_1, then the generalized representation condition (3) holds for F at $\overline{x}$ with respect to M_2.*

The following propositions summary properties of the generalized representation condition (3) (see Yang (1998)) and will be used in the sequel to establish solution characterizations for quadratic programming and linear fractional programming problems.

PROPOSITION 2.2. *Any quadratic function satisfies the generalized representation condition (3).*

Proof. See Example 3.3 in Yang (1998).

PROPOSITION 2.3. *Any linear fractional function satisfies the generalized representation condition (3).*

Proof. Let

$$f(x) = \frac{a^T x + r}{b^T x + s}$$

where $a, b, x \in \mathbb{R}^n$ and $r, s \in R$ satisfying $b^T x + s > 0$. From Yang (1998), we have

$$f(x) = f(\overline{x}) + \nabla f(\overline{x})^T (x - \overline{x}) + \frac{1}{2} \frac{b^T \overline{x} + s}{b^T x + s} (x - \overline{x})^T \nabla^2 f(\overline{x})(x - \overline{x}), \qquad x, \overline{x} \in D_f.$$

Let

$$\eta(x, \overline{x}) = \sqrt{\frac{b^T \overline{x} + s}{b^T x + s}} (x - \overline{x}). \tag{4}$$

Then the generalized representation condition (3) is satisfied.

3. A second-order global sufficient condition

In this section we obtain second-order optimality conditions of a constrained optimization problem where the objective function and the constraints are compositions of convex functions and twice strictly differentiable functions by transforming it into a convex composite optimization problem.

Let C be a closed convex set of $\mathbb{R}^n$, $g_i : \mathbb{R}^k \longrightarrow \mathbb{R}$, $i = 0, 1, \cdots, m$ be convex functions, $F_i = (F_{i1}, \cdots, F_{ik})^T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, $i = 0, 1, \cdots, m$ be twice strictly differentiable functions. Let

$$A = \{x : x \in C, g_i(F_i(x)) \leqslant 0, i = 1, \cdots, m\},$$

Consider the following optimization problem

$$\text{(P1)} \qquad \text{Minimize } g_0(F_0(x)) \quad \text{subject to } x \in A.$$

The first-order optimality conditions for the problem (P1) where F_i is Gateaux differentiable only have been derived in Jeyakumar and Yang (1993).

The **Slater** constraint qualification of (P1) is said to hold if

$$\exists x_0 \in intC, g_i(F_i(x_0)) < 0, i = 1, \cdots, m. \tag{5}$$

Problem (P1) is said to be calm at a point $\overline{x}$ if there exists $M > 0$ such that for any $u_k = (u_{k,1}, \ldots, u_{k,m}) \in \mathbb{R}^m_+$ with $\|u_k\| \to 0^+$ (namely, $\|u_k\| \neq 0$ and $\|u_k\| \to 0$), for any x_k satisfying $g_i(F_i(x_k)) \leqslant u_{k,i}, i = 1, \ldots, m$ and $x_k \to \overline{x}$, there holds

$$\frac{\bar{f}(x_k) - \bar{f}(\overline{x})}{\|u_k\|} + M \geqslant 0, \forall k,$$

where $\bar{f} = g_0(F_0(x))$.

From Corollary 2 in page 238 of Clarke (1983) and from Proposition 6.4.2 again in Clarke (1983) it follows that if C is bounded and (5) holds, then the problem (P1) is calm at a minimizer. Suppose that $\overline{x}$ is a minimizer of (P1) and the problem (P1) is calm at $\overline{x}$. By Proposition 6.4.3 in Clarke (1983), there exists $M > 0$ such that $\overline{x}$ is a minimizer of the optimization problem

$$\text{Minimize } f_0(x) \quad \text{subject to } x \in C,$$

where $f_0(x) = g_0(F_0(x)) + M \sum_{i=1}^m \max\{0, g_i(F_i(x))\}$.

Let

$$D = C \times \mathbb{R}^k \times \mathbb{R}^k \times \cdots \times \mathbb{R}^k, \tag{6}$$

$$g(z) = g(x, z_0, z_1, \cdots, z_m) = \begin{cases} g_0(z_0) + M \sum_{i=1}^m \max\{0, g_i(z_i)\}, & z \in D, \\ +\infty, & \text{otherwise}, \end{cases} \tag{7}$$

$$F(x) = (x^T, F_0(x)^T, F_1(x)^T, \cdots, F_m(x)^T)^T, \qquad x \in \mathbb{R}^n. \tag{8}$$

Then $f_0(x) = g(F(x))$. From the convexity of g_i and the monotonicity of $m(r) = \max\{0, r\}$ as a function of r, it is easy to see that g is a lower semi-continuous convex function with $dom(g) = D$. The function F is twice strictly differentiable. Then the problem (P1) is formulated as a convex composite optimization problem (CP) with g and F defined by (6)-(8). This composite problem is denoted by (CPP). Let $K_{CPP}(x)$ be the critical cone of (CPP). We have $\nabla F(x) = (I_{n\times n}, \nabla F_0(x)^T, \nabla F_1(x)^T, \cdots, \nabla F_m(x)^T)^T$.

LEMMA 3.1. *Let $\overline{x} \in A$. Then*

$$K_{CPP}(\overline{x}) = \{u \in cone(C-\overline{x}) : \forall i \in I(\overline{x}) \cup \{0\},$$
$$\exists v_i \in \partial g_i(F_i(\overline{x})), v_i^T \nabla F_i(\overline{x})u \leqslant 0\}, \tag{9}$$

where $I(\overline{x}) = \{i : g_i(F_i(\overline{x})) = 0, i = 1, \cdots, m\}$ is the active constraint index set.

Proof. The vector $u \in K_{CPP}(\overline{x})$ if and only if

$$g(F(\overline{x}) + t\nabla F(\overline{x})u) \leqslant g(F(\overline{x})), \quad \text{forsome } t > 0,$$

thus

$$g(\overline{x} + tu, F_0(\overline{x}) + t\nabla F_0(\overline{x})u, F_1(\overline{x}) + t\nabla F_1(\overline{x})u, \cdots,$$
$$F_m(\overline{x}) + t\nabla F_m(\overline{x})u) \leqslant g(\overline{x}, F_0(\overline{x}), F_1(\overline{x}), \cdots, F_m(\overline{x})).$$

Then $u \in cone(C-\overline{x})$ and

$$g_0(F_0(\overline{x}) + t\nabla F_0(\overline{x})u) - g_0(F_0(\overline{x}))$$
$$+ M\sum_{i=1}^{m} \max\{0, g_i(F_i(\overline{x}) + t\nabla F_i(\overline{x})u)\} \leqslant 0, \quad \text{for some } t > 0.$$

This is equivalent to

$$u \in cone(C-\overline{x}),$$
$$g_0(F_0(\overline{x}) + t\nabla F_0(\overline{x})u) - g_0(F_0(\overline{x})) \leqslant 0, \tag{10}$$
$$\max\{0, g_i(F_i(\overline{x}) + t\nabla F_i(\overline{x})u)\} \leqslant 0, \qquad \forall i \in I(\overline{x}). \tag{11}$$

From the convexity of g_0, there exists $v_0 \in \partial g_0(F_0(\overline{x}))$ such that

$$tv_0^T \nabla F_0(\overline{x})u \leqslant g_0(F_0(\overline{x}) + t\nabla F_0(\overline{x})u) - g_0(F_0(\overline{x})).$$

Thus (10) is equivalent to

$$v_0^T \nabla F_0(\overline{x})u \leqslant 0, \qquad \exists v_0 \in \partial g_0(F_0(\overline{x})).$$

By the same arguments, (11) is equivalent to

$$v_i^T \nabla F_i(\overline{x})u \leqslant 0, \qquad \exists v_i \in \partial g_i(F_i(\overline{x})), \forall i \in I(\overline{x}).$$

Hence (9) holds. □

LEMMA 3.2. *Let $\overline{x} \in A$. The following holds*

$$\partial g(F(\overline{x})) = N(\overline{x}|C) \times \partial g_0(F_0(\overline{x})) \times \Pi_{i\notin I(\overline{x})}\{0\} \times \Pi_{i\in I(\overline{x})}\{M\alpha_i y_i^* : y_i^* \in \partial g_i(F_i(\overline{x})), \alpha_i \in [0,1]\}, \tag{12}$$

where $\times$ is the product of sets in a product space. The set of optimal multipliers for (CPP) is

$$\begin{aligned} L_0^{CPP}(\overline{x}) = \{y^* : y^* = (&y_c^{*T}, y_0^{*T}, M\alpha_1 y_1^{*T}, \cdots, M\alpha_m y_m^{*T})^T, \\ &y_c^* + \nabla F_0(\overline{x})^T y_0^* + M \sum_{i\in I(\overline{x})} \alpha_i \nabla F_i(\overline{x})^T y_i^* = 0, \\ &y_c^* \in N(\overline{x}|C);\ y_i^* = 0,\ i \notin I(\overline{x}); \\ &y_i^* \in \partial g_i(F_i(\overline{x})), \alpha_i \in [0,1], i \in I(\overline{x}) \cup \{0\}\}. \end{aligned} \tag{13}$$

Proof. The proof of (11) is standard convex analysis arguments and omitted. (12) follows from the definition of $L_0(x)$. □

Using Lemmas 3.1 and 3.2, second-order optimality conditions of (P1) are established.

THEOREM 3.1. *Consider the problem (P1).*

(i) *If (P1) is calm at $\overline{x}$ and $\overline{x}$ is a minimizer of (P1), then $L_0^{CPP}(\overline{x}) \neq \emptyset$ and*

$$\max\{\sum_{j=1}^{k} y_{0j}^* u^T D^2 F_{0j}(\overline{x}) u + m \sum_{i\in I(\overline{x})} \alpha_i \sum_{j=1}^{k} y_{ij}^* u^T D^2 F_{ij}(\overline{x}) u : y_i^* \in \partial g_i(F_i(\overline{x}))\} \geqslant 0, \quad \forall u \in \overline{K_{CPP}(\overline{x})}, \tag{14}$$

(ii) *If $L_0^{CPP}(\overline{x}) \neq \emptyset$, the generalized representation condition (3) holds for F_i at $\overline{x}$ with respect to $\overline{K_{CPP}(\overline{x})}$ with the same $\eta(x, \overline{x})$, $i \in I(\overline{x}) \cup \{0\}$ and (13) holds, then $\overline{x}$ is a minimizer of (P1).*

Proof. (i) Since the problem (P1) is calm at $\overline{x}$, $\overline{x}$ is a minimizer of (CPP). From Lemma 3.2 in Yang (1998), the convex composite optimization problem (CPP) satisfies the basic constraint qualification. Then $L_0^{CPP}(\overline{x}) \neq \emptyset$. For $y^* \in L_0^{CPP}(\overline{x})$, $L(x, y^*) = \langle y_c^*, x\rangle + \langle y_0^*, F_0(x)\rangle + m \sum_{i\in I(x^*)} \alpha_i \langle y_i^*, F_i(x)\rangle$. Then

$$L^{\circ\circ}(\overline{x}, y^*; u, u) = \sum_{j=1}^{k} y_{0j}^* u^T D^2 F_{0j}(\overline{x}) u + m \sum_{i\in I(\overline{x})} \alpha_i \sum_{j=1}^{k} y_{ij}^* u^T D^2 F_{ij}(\overline{x}) u.$$

Thus (i) follows from Lemma 3.1 and Theorem 2.1 (ii).

(ii) For any feasible minimizer $x \in A$, we have for $y^* \in L_0^{CPP}(\overline{x})$,

$$g(F(x)) - g(F(\overline{x})) \geqslant \langle y^*, F(x) - F(\overline{x})\rangle. \tag{15}$$

Noting that $y^* = (y_c^{*T}, y_0^{*T}, M\alpha_1 y_1^{*T}, \cdots, M\alpha_m y_m^{*T})^T$, where

$$y_c^* + \nabla F_0(\overline{x})^T y_0^* + m \sum_{i \in I(x^*)} \alpha_i \nabla F_i(\overline{x})^T y_i^* = 0,$$

$y_c^* \in N(\overline{x}|C)$; $y_i^* = 0, i \notin I(\overline{x})$; $y_i^* \in \partial g_i(F_i(\overline{x})), \alpha_i \in [0, 1], i \in I(\overline{x}) \cup \{0\}$, thus

$$\begin{aligned}
&\langle y^*, F(x) - F(\overline{x})\rangle \\
&= \langle y_c^*, x - \overline{x}\rangle + \langle y_0^*, F_0(x) - F_0(\overline{x})\rangle + \sum_{i \in I(\overline{x})} M\alpha_i \langle y_i^*, F_i(x) - F_i(\overline{x})\rangle \\
&= \langle y_c^*, x - \overline{x}\rangle \\
&\quad + \langle \nabla F_0(\overline{x})^T y_0^*, x - \overline{x}\rangle + \langle y_0^*, \eta(x, x^*)^T D^2 F_0(x^*) \eta(x, x^*)\rangle \\
&\quad + \sum_{i \in I(\overline{x})} M\alpha_i \langle \nabla F_i(\overline{x})^T y_i^*, x - \overline{x}\rangle \\
&\quad + \sum_{i \in I(\overline{x})} \alpha_i \langle y_i^*, \eta(x, \overline{x})^T D^2 F_i(x^*) \eta(x, \overline{x})\rangle \\
&= \langle y_c^* + \nabla F_0(\overline{x})^T y_0^* \\
&\quad + m \sum_{i \in I(\overline{x})} \alpha_i \nabla F_i(\overline{x})^T y_i^*, x - \overline{x}\rangle + \\
&\quad \sum_{j=1}^{k} y_{0j}^* \eta(x, \overline{x})^T D^2 F_{0j}(\overline{x}) \eta(x, \overline{x}) \\
&\quad + m \sum_{i \in I(\overline{x})} \alpha_i \sum_{j=1}^{k} y_{ij}^* \eta(x, \overline{x})^T D^2 F_{ij}(\overline{x}) \eta(x, \overline{x}) \\
&= \sum_{j=1}^{k} y_{0j}^* \eta(x, \overline{x})^T D^2 F_{0j}(\overline{x}) \eta(x, \overline{x}) \\
&\quad + m \sum_{i \in I(\overline{x})} \alpha_i \sum_{j=1}^{k} y_{ij}^* \eta(x, \overline{x})^T D^2 F_{ij}(\overline{x}) \eta(x, \overline{x}),
\end{aligned}$$

where

$$\eta(x, \overline{x})^T D^2 F_i(\overline{x}) \eta(x, \overline{x}) = \begin{pmatrix} \eta(x, \overline{x})^T D^2 F_{i1}(\overline{x}) \eta(x, \overline{x}) \\ \vdots \\ \eta(x, \overline{x})^T D^2 F_{ik}(\overline{x}) \eta(x, \overline{x}) \end{pmatrix},$$

$$\eta(x,\overline{x}) \in \overline{K_{CCP}(\overline{x})}.$$

Then from (14), for every $x \in A$,

$$\begin{aligned} & g(F(x)) - g(F(\overline{x})) \\ & \geqslant \langle y^*, F(x) - F(\overline{x})\rangle \\ & \geqslant \sum_{j=1}^{k} y_{0j}^* \eta(x,\overline{x})^T D^2 F_{0j}(\overline{x})\eta(x,\overline{x}) \\ & \quad + m \sum_{i\in I(\overline{x})} \alpha_i \sum_{j=1}^{k} y_{ij}^* \eta(x,\overline{x})^T D^2 F_{ij}(\overline{x})\eta(x,\overline{x}) \\ & \geqslant 0. \end{aligned}$$

Then $\overline{x}$ is a minimizer of (CPP), so is a minimizer of (P1). □

4. Second-order minimizer characterizations of a convex program

This section derives second-order characterizations of the minimizer set for a convex program using a generalized representation condition. Consider the convex program

$$\text{(P2)} \qquad \text{Minimize } f(x) \qquad \text{subjectto } x \in C,$$

where C is a convex set of $\mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a convex function. The following first order characterization of (P2) was obtained in Mangasarian (1988).

LEMMA 4.1. *Mangasarian* (1988) *If f is differentiable, then the minimizer set $\overline{C}$ of (P2) is characterized by*

$$\overline{C} = \{x \in C : f(x) = f(\overline{x}), \langle \nabla f(\overline{x}), x - \overline{x}\rangle = 0\},$$

where $\overline{x}$ is a minimizer of $(P2)$.

We now obtain a second-order characterization of the minimizer set for *(P2)* using a generalized representation condition.

THEOREM 4.1. *Assume that f is twice strictly differentiable and satisfies the generalized representation condition (3). Then*

$$\overline{C} = \{x \in C : \langle \nabla f(\overline{x}), x - \overline{x}\rangle = 0, \langle D^2 f(\overline{x})(\eta(x,\overline{x})), \eta(x,\overline{x})\rangle = 0\}. \quad (16)$$

where $\overline{x}$ is a minimizer of (P2) and $\overline{C}$ is the set of minimizers.

Proof. Let

$$C_1 = \{x \in C : \langle \nabla f(\overline{x}), x - \overline{x} \rangle = 0, \langle D^2 f(\overline{x})(\eta(x, \overline{x})), \eta(x, \overline{x}) \rangle = 0\}$$

Suppose $x \in \overline{C}$. Then it follows from Lemma 4.1 that $f(x) = f(\overline{x})$, $\langle \nabla f(\overline{x}), x - \overline{x} \rangle = 0$. Using (3), we have

$$f(x) = f(\overline{x}) + \langle \nabla f(\overline{x}), x - \overline{x} \rangle + \frac{1}{2} \langle D^2 f(\overline{x})(\eta(x, \overline{x})), \eta(x, \overline{x}) \rangle.$$

Thus

$$\langle D^2 f(\overline{x})(\eta(x, \overline{x})), \eta(x, \overline{x}) \rangle = 0.$$

Then $x \in C_1$. The converse is trivial by (3). □

Consider the convex quadratic program

$$\text{(P3)} \qquad \text{Minimize } \; f(x) = \frac{1}{2} x^T Q x + p^T x$$
$$\text{subjectto } \; x \in C,$$

where Q is a symmetric positive semi-definite matrix and $p \in \mathbb{R}^n$. Recently in Burke and Ferris (1993) the following second-order characterization of a minimizer was obtained for the convex quadratic program (P3):

$$\overline{C} = \{x \in C : \langle Q\overline{x} + p, x - \overline{x} \rangle = 0, Q(x - \overline{x}) = 0\}. \tag{17}$$

where $\overline{x}$ is a minimizer of (P3).

COROLLARY 4.1. *Consider the convex quadratic program. Then*

$$\overline{C} = \{x \in C : \langle Q\overline{x} + p, x - \overline{x} \rangle = 0, (x - \overline{x})^T Q (x - \overline{x}) = 0\}, \tag{18}$$

where $\overline{x}$ is a minimizer of (P3).

It can be easily proved that (17) and (18) are equivalent by employing the following result (see Eisenberg, 1962): for an $n \times n$ symmetric positive semi-definite matrix Q and $x \in \mathbb{R}^n$,

$$x^T Q x = 0$$

if and only if

$$Qx = 0.$$

5. Second-order minimizer characterizations of a linear fractional program

In this section we present some second order minimizer characterization for the following linear fractional program

$$\text{(P4)} \qquad \text{Minimize } f(x) = \frac{a^T x + r}{b^T x + s}$$
$$\text{subjectto } x \in D_f,$$

where $a, b, x \in R^n$ and $r, s \in R$, and $D_f = \{x \in R^n | b^T x + s > 0\}$.

By Jeyakumar and Yang (1995), taking into account a characterization of the solution set obtained for pseudolinear function, the following result easily follows.

LEMMA 5.1. *Jeyakumar and Yang* (1995) *The minimizer set* $\overline{D_f}$ *of* (*P*4) *is characterized by*

$$\overline{D_f} = \{x \in D_f : \langle \nabla f(\overline{x}), x - \overline{x} \rangle = 0\}, \tag{19}$$

where $\overline{x}$ *is a minimizer of (P4).*

For linear fractional functions simple calculation establishes that

$$\nabla f(\overline{x}) = \frac{a(b^T\overline{x} + s) - b(a^T\overline{x} + r)}{(b^T\overline{x} + s)^2}$$

$$\nabla^2 f(\overline{x}) = \frac{-(a^T\overline{x} + r)(ab^T + ba^T) + 2(b^T\overline{x} + t)aa^T}{(a^T\overline{x} + r)^3}.$$

By Theorem 4.1 we obtain.

THEOREM 5.1. *Consider the linear fractional program (P4). Then*

$$\overline{D_f} = \{x \in D_f : [(b^T\overline{x})a^T - (a^T\overline{x})b^T]x = (sa - rb)^T(\overline{x} - x),$$
$$[(a^T x + r)b^T - (b^T x + t)a^T](x - \overline{x}) = 0\}$$

where $\overline{x}$ *is a minimizer of (P4).*

Proof. It follows from Theorem 4.1 and the generalized representation condition (4) that

$$\begin{aligned}\overline{D_f} &= \{x \in D_f : \langle \nabla f(\overline{x}), x - \overline{x} \rangle = 0, \langle \nabla^2 f(\overline{x})(\eta(x, \overline{x})), \eta(x, \overline{x}) \rangle = 0\} \\ &= \{x \in D_f : \langle \nabla f(\overline{x}), x - \overline{x} \rangle = 0, (x - \overline{x})^T \nabla^2 f(\overline{x})(x - \overline{x}) = 0\}. \end{aligned} \tag{20}$$

Thus, $\langle \nabla f(\overline{x}), x - \overline{x} \rangle = 0$ is equivalent to

$$[(b^T\overline{x})a^T - (a^T\overline{x})b^T]x = (sa - rb)^T(\overline{x} - x)$$

and $(x - \overline{x})\nabla^2 f(\overline{x})(x - \overline{x}) = 0$ is equivalent to

$$(a^T x + r)(x - \overline{x})^T ab^T (x - \overline{x}) = (b^T x + t)(a^T (x - \overline{x}))^2$$

or

$$[(a^T x + r)b^T - (b^T x + t)a^T](x - \overline{x}) = 0.$$

The conclusion holds. □

The following characterization was derived in Jeyakumar and Yang (1995) for twice continuously differentiable pseudolinear functions,

$$\begin{aligned}\overline{D_f} = \{x \in D_f : \langle \nabla f(x\alpha\overline{x}), x - \overline{x}\rangle = 0,\ \forall \alpha \in [0, 1]\}\\ \cap\{x \in D_f : (x - \overline{x})^T \nabla^2 f(x\alpha\overline{x})(x - \overline{x}) = 0,\ \forall \alpha \in [0, 1]\}, \qquad (21)\end{aligned}$$

(where $x\alpha\overline{x} = \alpha x + (1 - \alpha)\overline{x}$), using a second order characterization of a pseudolinear function obtained in Chew and Choo (1984). It is clear that (21) includes (20) as a special case. However, the pseudolinear function $f(x) = x + \frac{1}{2}\sin(x)$ does not satisfy the generalized representation condition at $\overline{x} = 0$. Hence, Theorem 5.1 gave an alternative proof for the second order characterization of a minimizer for a linear fractional program.

6. Conclusion

We studied a second-order generalized representation condition. This condition is applied to derive a global second-order sufficient condition for an optimization problem with a convex composite objective and convex composite inequality constraints and to characterize the minimizer of a convex program and a linear fractional program.

References

1. Burke, J.V. and Ferris, M.C. (1993), Weak sharp minima in mathematical programming, *SIAM Journal on Control and Optimization* 31, 1340–1359.
2. Burke, J.V. and Poliquin, R.A. (1992), Optimality conditions for non-finite valued convex composite functions, *Mathematical Programming* 57, 103–120.
3. Chew, K.L. and Choo, E.V. (1984), Pseudolinear and efficiency, *Mathematical Programming* 28, 226–239.
4. Clarke, F.H. (1983), *Optimization and Nonsmooth Analysis*, John Wiley, New York.
5. Cominetti, R. and Correa, R. (1990), A Generalized second-order derivative in nonsmooth optimization, SIAM *Journal on Control and Optimization* 28, 789–809.
6. Eisenberg, E. (1962), Support of a convex function, *Bulletin of the American Mathematical Society* 68, 192–195.

7. Hiriart-Urruty, J.B., Strodiot, J.J. and Hien Nguyen, V. (1984), Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data, *Applied Mathematics and Optimization* 11, 43–56.
8. Horst, R. and Tuy, H. (1990), Global Optimization: Deterministic Approaches, Springer, Berlin, New York.
9. Ioffe, A.D. (1979), Necessary and sufficient conditions for a local minimum. 2: Conditions of Levitin-Miljutin-Osmolovskii type, *SIAM Journal on Control and Optimization* 17, 251–265.
10. Jeyakumar, V. and Yang, X.Q. (1993), Convex composite multi-objective nonsmooth programming, mathematical programming 59, 325–343.
11. Jeyakumar, V. and Yang, X.Q. (1995), On characterizing the solution sets of pseudolinear programs, *Journal of Optimization Theory and Applications* 87, 747–755.
12. Mangasarian, O.L. (1988), A simple characterization of solution sets of convex programs, *Operations Research Letters* 7, 21–26.
13. Rockafellar, R.T. (1988), First- and second-order epi-differentiability in nonlinear programming, *Transactions of the American Mathematical Society* 307, 75–108.
14. Yang, X.Q. (1998), Second-order global optimality conditions of convex composite optimization, *Mathematical Programming* 81, 327–347.
15. Yang, X.Q. and Jeyakumar, V. (1992), Generalized second-order directional derivatives and optimization with $C^{1,1}$ functions, *Optimization* 26, 165–185.